

A Sampling Theorem for Rotation Numbers of Linear Processes in \mathbf{R}^2

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Abstract

We prove an ergodic theorem for the rotation number of the composition of a sequence of stationary random homeomorphisms in S^1 . In particular, the concept of rotation number of a matrix $g \in Gl^+(2, \mathbf{R})$ can be generalized to a product of a sequence of stationary random matrices in $Gl^+(2, \mathbf{R})$. In this particular case this result provides a counter-part of the Osseledec's multiplicative ergodic theorem which guarantees the existence of Lyapunov exponents. A random sampling theorem is then proved to show that the concept we propose is consistent by discretization in time with the rotation number of continuous linear processes on \mathbf{R}^2 .

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1 Introduction

The rotation number of an orientation-preserving homeomorphism in the circle $f : S^1 \rightarrow S^1$ describes the average rotation that f does in S^1 , either if

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we consider this average in time for the orbit of a given starting point or the average on the angular displacement with respect to an f -invariant measure on S^1 (cf. Corollary 3.3). Considering the radial projection $\mathbf{R}^2 \setminus \{0\} \rightarrow S^1$, this concept extends naturally to the space of linear operators on \mathbf{R}^2 with positive determinant $Gl^+(2, \mathbf{R})$. So, on one hand the Lyapunov exponents give the asymptotic exponential rate of increasing in the radial coordinate, on the other hand, rotation number completes the information about the long time behaviour of the system giving the asymptotic angular behaviour.

Let g be a matrix in $Gl^+(2, \mathbf{R})$, when we write its eigenvalues as

$$\lambda_1 = e^{a+bi} \quad \text{and} \quad \lambda_2 = e^{a'-bi} ,$$

with $a = a'$ if $b \in (0, \pi)$, the Lyapunov spectrum is $\{a, a'\}$ and the rotation number is

$$\rho(g) = \pm \frac{b}{2\pi} ,$$

with the sign depending on the orientation chosen in the plane (see Proposition 4.2). This twined relation between Lyapunov exponents and rotation numbers suggests the questions:

- i) Is there any extension of the concept of rotation number for the product of a sequence of stationary random matrices in $Gl^+(2, \mathbf{R})$?
- ii) If it does, can we assure its existence almost surely like Oseledec's multiplicative ergodic theorem does for their Lyapunov exponents?
- iii) Is this concept consistent by discretization in time (sampling) with the rotation number of continuous linear processes in \mathbf{R}^2 ?

The main purpose of this paper is to prove that the answers to these questions are affirmative. In section 2 we review the classical definition of rotation number and we give an alternative interpretation in terms of *ordered lifted orbit* of a point p in S^1 (Proposition 2.1). In section 3 we extend the definition of rotation numbers for a sequence of orientation-preserving homeomorphisms in S^1 , and although the main interest in this paper is rotation number for linear processes, we prove an ergodic theorem of existence of these numbers in a more general context: for the composition of a sequence of stationary random orientation-preserving homeomorphisms in S^1 . In section 4 we study the case of a product of random matrices. A random sampling theorem is then proved in section 5 to show that the concept we propose is consistent with the rotation number of continuous linear cocycles in \mathbf{R}^2 .

2 Classical Definition

Let $Hom^+(S^1)$ denote the set of homeomorphisms which preserve orientation in the circle S^1 . Given $f \in Hom^+(S^1)$, we say that a continuous monotone increasing function $F : \mathbf{R} \rightarrow \mathbf{R}$ is a lift of f if $f \circ \pi = \pi \circ F$, where π is the canonical projection $\pi : \mathbf{R} \rightarrow S^1$, $x \mapsto e^{2\pi xi}$. Two lifts of the same homeomorphism will differ by an integer translation. Given a lift F of f then $F(x+1) = F(x) + 1$, which implies that $(F - Id)$ is periodic with period 1 and

$$\max_{x \in \mathbf{R}} \{(F - Id)(x)\} - \min_{x \in \mathbf{R}} \{(F - Id)(x)\} < 1.$$

We set $F^{(n)} = F \circ F \circ \dots \circ F$, n -times. The classical definition of rotation number of f is, then:

$$\rho(f) = \lim_{n \rightarrow \infty} \frac{F^{(n)}(x) - x}{n} \pmod{1} \quad (1)$$

and it is well known that this limit exists and is independent of the initial point x and of the lift F chosen. The homeomorphism $f : S^1 \rightarrow S^1$ has periodic point if and only if $\rho(f) \in \mathbf{Q}$, and if $\rho(f) \in \mathbf{R} \setminus \mathbf{Q}$ then f is semi-conjugate with $R_{\rho(f)}$, the rotation by the angle $\rho(f)$ (see *e.g.* Nitecki[6]).

Given a point $p \in S^1$ and $x \in (-1/2, 1/2]$ such that $\pi(x) = p$, we construct inductively the following real, monotone non-decreasing sequence: $\theta_{p,0} = x$ and $\theta_{p,n}$ is the smallest real number such that $\theta_{p,n} \geq \theta_{p,n-1}$ and $\pi(\theta_{p,n}) = f^{(n)}(p)$. The sequence $\{\theta_{p,n}\}_{n \geq 0}$ is the ordered orbit of p in the covering space \mathbf{R} , it will be called the *ordered lifted orbit* of p . Note that $0 \leq \theta_{p,n} - \theta_{p,n-1} < 1$ for all positive integer n .

Proposition 2.1. *The rotation number $\rho(f)$ is the asymptotic average rate of increase of the ordered lifted orbit of p , independently of p chosen, i.e.,*

$$\rho(f) = \lim_{n \rightarrow \infty} \frac{\theta_{p,n} - \theta_{p,0}}{n} \pmod{1}. \quad (2)$$

Proof: We assume firstly that f has no fixed point in S^1 . Without loss of generality, take the unique lift F such that $0 < (F - Id)(x) < 1$ for all $x \in \mathbf{R}$, then, since F is strictly increasing, we have by construction that $\theta_{p,n} = F^{(n)}(x)$ for all $n \geq 0$, which implies equation (2).

Suppose now that f has at least one fixed point in S^1 , hence $\rho(f) = 0$. Take the unique lift F which has fixed point in \mathbf{R} . If $F(x) \geq x$, then again $\theta_{p,n} = F^{(n)}(x)$ and this sequence converges to the fixed point of F which is

the nearest from x by above. So, the limit in equation (2) vanishes and the equality holds. Finally, if $F(x) < x$, let y be the fixed point of F which is the nearest from x by below, then $F^{(n)}(x)$ is a strictly decreasing sequence converging to y and $\theta_{p,1} = F(x) + 1, \dots, \theta_{p,n} = F(\theta_{p,n-1}) + 1$, hence

$$\lim_{n \rightarrow \infty} \frac{\theta_{p,n} - \theta_{p,0}}{n} = \lim_{n \rightarrow \infty} \frac{F^{(n)}(x) + n - x}{n} = 0 \pmod{1}.$$

Moreover, since the rotation number is independent of the initial point x , then the limit in equation (2) is also independent of p . □

REMARK: If the homeomorphisms $f, g : S^1 \rightarrow S^1$ commute then $\rho(f \circ g) = \rho(f) + \rho(g) \pmod{1}$.

3 Rotation Number for Composition of Random Homeomorphisms in the Circle

Now consider a sequence of orientation-preserving homeomorphisms $f_1, f_2, \dots : S^1 \rightarrow S^1$. What we have seen in the last section suggests to us two approaches to extend the concept of rotation number for the composition of these homeomorphisms $(f_n \circ f_{n-1} \circ \dots \circ f_1)_{n \geq 1}$:

THE FIRST APPROACH. To consider the composition of a sequence of lifts $f_1^\sim, f_2^\sim, \dots$ of f_1, f_2, \dots , respectively. For the same reason that in the definition of rotation number for a single homeomorphism f with a lift F , when we take the composition $F^{(n)} = F \circ \dots \circ F$ we can not make this composition with distinct lifts, say $F_1 \circ F_2 \dots \circ F_n$ with $(F_j)_{j \geq 1}$ a sequence of distinct lifts of f , here, in order to have the same kind of compatibility among the lifts $f_1^\sim, f_2^\sim, \dots$, we will state that all of them start in the same interval, say $f_j^\sim(0) \in (-1/2, 1/2]$ for all j positive integer. Set $F_n^\sim = f_n^\sim \circ \dots \circ f_1^\sim$ and given $x \in \mathbf{R}$, define:

$$\rho(f_1, f_2, \dots) = \lim_{n \rightarrow \infty} \frac{F_n^\sim(x) - x}{n} \pmod{1}, \quad (3)$$

when the limit exists.

THE SECOND APPROACH. Define the rotation number of the composition $f_n \circ \dots \circ f_1 : S^1 \rightarrow S^1$, $n = 1, 2, \dots$ via the ordered lifted orbit of a point $p \in S^1$. Namely if $x \in (-1/2, 1/2]$ such that $\pi(x) = p$ then construct

inductively the sequence: $\theta_{p,0} = x$, and $\theta_{p,n}$ is the smallest real number such that $\theta_{p,n-1} \leq \theta_{p,n}$ and $\pi(\theta_{p,n}) = f_n \circ \dots \circ f_1(p)$, and define

$$\text{rot}(f_1, f_2, \dots)(p) = \lim_{n \rightarrow \infty} \frac{\theta_{p,n} - \theta_{p,0}}{n} \pmod{1} \quad (4)$$

when the limit exists.

Next proposition shows that the first approach is particularly interesting for our purposes.

Proposition 3.1. *If the rotation number $\rho(f_1, f_2, \dots)$ of equation (3) exists then it is independent of the initial point $x \in \mathbf{R}$.*

Proof: The proof goes similarly to the proof for the single homeomorphism case. Arguing inductively, F_n^\sim as defined above is a lift of $f_n \circ \dots \circ f_1$, hence, for all positive integer n :

$$\max_{x \in \mathbf{R}} \{(F_n^\sim - Id)(x)\} - \min_{x \in \mathbf{R}} \{(F_n^\sim - Id)(x)\} < 1,$$

then,

$$|F_n^\sim x - F_n^\sim y| \leq |(F_n^\sim x - x) - (F_n^\sim y - y)| + |x - y| \leq |x - y| + 1,$$

therefore

$$\lim_{n \rightarrow \infty} \left(\frac{F_n^\sim x}{n} - \frac{F_n^\sim y}{n} \right) = 0;$$

hence if the limit of equation (3) exists, it is independent of x . □

EXAMPLE 1. For simplicity we parametrize S^1 by $x \mapsto e^{2\pi xi}$ with $x \in (-1/2, 1/2]$. Let f_1, f_2, f_3 and f_4 be such that $f_i(0) = 0$, $i=1,2,3,4$, and $f_1(1/8) = 3/8$, $f_2(3/8) = -3/8$, $f_3(-3/8) = -1/8$ and $f_4(-1/8) = 1/8$. Consider the sequence f_1, f_2, \dots such that $f_n = f_{(n \bmod 4)}$. For the first approach $\rho(f_1, f_2, \dots) = 0$ since 0 is a fixed point of $f_n \circ \dots \circ f_1$. For the second approach $\text{rot}(f_1, f_2, \dots)(0) = 0$, nevertheless, if $p = 1/8$, then

$$\theta_{p,n} = \frac{2n+1}{8},$$

which yields $\text{rot}(f_1, f_2, \dots)(p) = 1/4$.

This example shows not only that the two approaches lead to different numbers but also that in the second one this number depends on the starting point p . From now on, whenever we refer to rotation number, we will

mean the first approach.

REMARK: Proposition 2.1 shows that for a single homeomorphism f , the two approaches coincide.

Once stated the compatibility condition that any lift F will be taken with $F(0) \in (-1/2, 1/2]$, we can set a metric in $Hom^+(S^1)$ given by the uniform metric in the lifts. This definition creates natural discontinuities at homeomorphisms f with lift $F(0) = 1/2$.

Although in this paper our main interest is rotation numbers for linear systems, an ergodic theorem of existence of such numbers can be stated in a more general context: for composition of a sequence of stationary random homeomorphisms in $Hom^+(S^1)$.

To introduce formally the set up, consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\theta : \Omega \rightarrow \Omega$ a measure-preserving transformation on Ω . Let us assume throughout for simplicity that θ is ergodic. Let $f : \Omega \rightarrow Hom^+(S^1)$ be a random homeomorphism in S^1 , we shall consider the sequence of stationary random homeomorphisms given by $f_n = f \circ \theta^{n-1}$. The main result of this section guarantees the existence \mathbb{P} -almost surely of the rotation number $\rho(f, \theta)$ of the discrete random dynamical system on S^1 over θ given by the composition $(f_n \circ f_{n-1} \circ \dots \circ f_1)_{n \geq 1}$.

For each $n \geq 1$, we will write the lift f_n^\sim as

$$f_n^\sim(\omega, x) = x + \delta_n(\omega, x) , \quad (5)$$

where the function $\delta_n(\omega, x)$ is periodic in the variable x with period 1 and $|\delta_n(\omega, x)| < 3/2$, for all x . In the sequel it will be convenient to write the δ_n 's as functions on S^1 , so for each n we define $\beta_n : \Omega \times S^1 \rightarrow (-3/2, 3/2)$ by

$$\beta_n(\omega, \pi(x)) = \delta_n(\omega, x) . \quad (6)$$

We denote by μ an invariant probability measure on $\Omega \times S^1$ for the skew product map $\Theta(\omega, s) = (\theta(\omega), f(\omega, s))$. The invariant measure μ factorizes as $\mu(ds, d\omega) = \nu_\omega(ds) \mathbb{P}(d\omega)$ (see, e.g. Crauel [4] or Arnold [1]).

With this set up we state the following ergodic theorem:

Theorem 3.2. *Consider the discrete random dynamical system on S^1 over θ given by $(f_n \circ f_{n-1} \circ \dots \circ f_1)_{n \geq 1}$ where $f_n = f \circ \theta^{n-1}$. Then, its rotation number $\rho(f, \theta)$ exists \mathbb{P} -a.s. and satisfies:*

$$\rho(f, \theta) = \mathbb{E} \left[\int_{S^1} \beta_1(\omega, s) d\nu_\omega(s) \right] \pmod{1} \quad a.s. \quad (7)$$

where $\mu = \nu_\omega(ds)\mathbb{P}(d\omega)$ is a (not necessarily ergodic) invariant probability measure on $\Omega \times S^1$ for the skew product map Θ .

Proof: By construction:

$$\beta_i(\omega, f_{i-1} \circ \dots \circ f_1(p)) = \beta_1 \circ \Theta^{i-1}(\omega, p) ,$$

for $i \geq 1$. Assume that $x \in \mathbf{R}$ is an initial point with $\pi(x) = p$, then by equation (5) and induction on n :

$$\begin{aligned} f_n^\sim \circ \dots \circ f_1^\sim(\omega, x) &= x + \delta_1(\omega, x) + \delta_2(\omega, x + \delta_1(\omega, x)) + \dots \\ &+ \delta_n(\omega, x + \delta_1(\omega, x) + \delta_2(\omega, x + \delta_1(\omega, x)) + \dots + \delta_{n-1}(\dots)) . \end{aligned}$$

Moreover, for any $i = 1, \dots, n$, by periodicity of δ_i (equation 6):

$$\begin{aligned} \delta_i(\omega, x + \delta_1(\omega, x) + \dots + \delta_{i-1}(\dots)) &= \delta_i(\omega, f_{i-1}^\sim \circ \dots \circ f_1^\sim(\omega, x)) \\ &= \beta_i(\omega, f_{i-1} \circ \dots \circ f_1(\omega, p)) \\ &= \beta_1 \circ \Theta^{i-1}(\omega, p) , \end{aligned} \tag{8}$$

hence:

$$f_n^\sim \circ \dots \circ f_1^\sim(\omega, x) = x + \sum_{i=1}^n \beta_1 \circ \Theta^{i-1}(\omega, p) .$$

If the measure μ is ergodic then by the Birkhoff's ergodic theorem:

$$\lim_{n \rightarrow \infty} \frac{1}{n} f_n^\sim \circ \dots \circ f_1^\sim(x) = \mathbb{E} \left[\int_{S^1} \beta_1(\omega, s) d\nu_\omega(s) \right] \quad \mu - a.s..$$

Note that once θ is ergodic on Ω , last formula says that there exists a subset $\Omega' \subset \Omega$ of probability one such that for each $\omega' \in \Omega'$, there exists $p \in S^1$ with the equality above holding for (ω', p) . The fact that the rotation number is independent of the initial point in S^1 (Proposition 3.1) implies that the equality also holds for (ω', s) for any $s \in S^1$. Hence, this dynamical restriction implies that the integral above does not depend on the ergodic measure chosen and can be taken with respect to any invariant probability measure, once they are convex combinations of the ergodic measures. \square

In particular, if the sequence $\{f_n\}_{n \geq 1}$ is i.i.d. then the process $(f_n \circ f_{n-1} \circ \dots \circ f_1(s))_{n \geq 1}$ in the circle S^1 is Markovian. In this case an invariant

measure μ is given by the product measure $\nu \otimes \mathbb{P}$ where ν is a stationary probability measure for the Markov process in S^1 . Hence, in this case the rotation number is given by:

$$\rho(f) = \int_{S^1} \mathbb{E} [\beta_1(\omega, s)] d\nu(s) \pmod{1} \quad a.s.. \quad (9)$$

We finish this section with a corollary whose proof follows naturally in this context, although it does not require a sequence of homeomorphisms. It shows that the rotation number of a homeomorphism f on S^1 is the average of angular displacement of the points in the circle with respect to any probability measure which is preserved by f . The proof comes directly from formula (9).

Corollary 3.3. *If $f \in \text{Hom}^+(S^1)$ preserves the probability measure ν on S^1 then*

$$\rho(f) = \int_{S^1} \beta(s) d\nu(s) \pmod{1},$$

where $\beta : S^1 \rightarrow \mathbf{R}$ is a continuous function on S^1 which gives a lift $F(x) = x + \beta(\pi(x))$ for f .

4 Product of Random Matrices in $Gl^+(2, \mathbf{R})$

We will denote by $\psi_g : S^1 \rightarrow S^1$ the action of the matrix $g \in Gl^+(2, \mathbf{R})$ over the circle S^1 , i.e. $\psi_g(x) = \frac{gx}{\|gx\|}$. The rotation number of ψ_g will be denoted simply by $\rho(g)$.

The next proposition shows that the rotation number of a matrix $g \in Gl^+(2, \mathbf{R})$ is a concept twined with its Lyapunov numbers in the sense that if $\lambda_1, \lambda_2 \in \mathbf{C}$ are the eigenvalues of g then the logarithm of their modulus $|\lambda_1|$ and $|\lambda_2|$ give the Lyapunov exponents and their arguments $\pm \arg(\lambda_1)$ give the rotation number. To prove this proposition we use the following lemma which states the invariance of the rotation number by conjugacy:

Lemma 4.1. *If $f, h \in \text{Hom}^+(S^1)$ then $\rho(h \circ f \circ h^{-1}) = \rho(f)$.*

Proof: Let H be a lift for h and consider its inverse $H^{-1} : \mathbf{R} \rightarrow \mathbf{R}$. Applying h^{-1} in both sides of $h \circ \pi = \pi \circ H$ we check that H^{-1} is also a lift for h^{-1} . For a lift F of f we have $(h \circ f \circ h^{-1}) \circ \pi = \pi \circ (H \circ F \circ H^{-1})$, i.e., $H \circ F \circ H^{-1}$ is a lift for $h \circ f \circ h^{-1}$ as well. The result follows immediately from definition:

$$\lim_{n \rightarrow \infty} \frac{(H \circ F \circ H^{-1})^n(x) - x}{n} = \lim_{n \rightarrow \infty} \frac{H \circ F^{(n)} \circ H^{-1}(x) - x}{n} = \rho(f) \pmod{1}$$

since $H(x) = Id(x) + \delta(x)$ where δ is a bounded periodic function. \square

REMARK: If we allow h to be a reverse-orientation homeomorphism in S^1 then its lift is a continuous monotone strictly decreasing functions, and it can be written as $H(x) = -Id(x) + \delta(x)$, for some periodic bounded function δ . So, in this case, the effect in the rotation number of the conjugacy by h is the change of the sign:

$$\rho(h \circ f \circ h^{-1}) = -\rho(f) \pmod{1}. \quad (10)$$

It corresponds of considering the rotation in the opposite (clockwise) direction.

Note that actually it does not matter in which interval we consider the rotation number $\rho(f)$ since it represents an equivalence class for the $\pmod{1}$ relation. However expressions like equation (10) would look more natural if we write $\rho(f)$ in the symmetric interval $(-1/2, 1/2]$. Another advantage of the representation of the equivalence classes in this interval is that it becomes easily comparable with rotation number for continuous process in S^1 via discretization, provided we take samples of the process at intervals of time $T > 0$ small enough (see Theorems 5.1 and 5.2).

Proposition 4.2. *Let g be a matrix in $Gl^+(2, \mathbf{R})$ and $\lambda_1, \lambda_2 \in \mathbf{C}$ be its eigenvalues. Then*

$$\rho(g) = \pm \frac{1}{2\pi} \arg(\lambda_1) \quad ,$$

where $\arg(\lambda_1) \in (-\pi, \pi]$ is the argument of the complex number λ_1 , and the sign will depend on the orientation chosen in \mathbf{R}^2 .

Proof: The proof comes immediately from Lemma 4.1: write g in its Jordan decomposition form $P\Lambda P^{-1}$, with $P \in Gl^+(2, \mathbf{R}^2)$; then $\rho(g) = \rho(\Lambda)$. \square

Considering the action of the group $Gl^+(2, \mathbf{R})$ on the circle S^1 , the concept of rotation number for the composition of a sequence of homeomorphisms is extended to the product of matrices in $Gl^+(2, \mathbf{R})$. By Theorem 3.2, for a stationary sequence of random matrices (in particular for i.i.d.) this number exists \mathbb{P} -almost surely, in the same way that the existence of the Lyapunov exponents in this case is assured by the Oseledec's multiplicative ergodic theorem (see *e.g.* Ruelle [9] or Furstenberg and Kifer [5] for a non-random filtration approach).

To set up the notation, let $Y : \Omega \rightarrow Gl^+(2, \mathbf{R})$ be a random matrix and $\theta : \Omega \rightarrow \Omega$ an ergodic transformation on Ω . We shall consider the sequence

of stationary random matrices given by $Y_n = Y \circ \theta^{n-1}$. Let $\Psi_{Y_n} : \mathbf{R} \rightarrow \mathbf{R}$ be the lift of $\psi_{Y_n} \in \text{Hom}^+(S^1)$, for each positive integer n . As before, we write

$$\Psi_{Y_n}(x) = x + \delta_n(\omega, x) \quad .$$

Note that here, because of the linearity of Y_n , the functions $\delta_n(\omega, x)$ has period $1/2$, also any invariant measure $\mu = \nu_\omega(ds)\mathbb{P}(d\omega)$ on $\Omega \times S^1$ can be considered such that ν_ω is a measure on the projective space $P\mathbf{R}^1$. We write $\beta_n(\omega, \pi(x)) = \delta_n(\omega, x)$, then by Theorem 3.2 the rotation number of the product of this sequence of random matrices exists and satisfies:

$$\rho(Y, \theta) = \mathbb{E} \left[\int_{S^1} \beta_1(\omega, s) d\nu_\omega(s) \right] \pmod{1} \quad a.s.. \quad (11)$$

In the i.i.d case (*cf.* formula (9)):

$$\rho(Y) = \int_{S^1} \mathbb{E} [\beta_1(\omega, s)] d\nu(s) \pmod{1} \quad a.s., \quad (12)$$

with ν a stationary probability measure on S^1 for the Markov process.

We present two simple but illustrative examples:

EXAMPLE 2. Let Y_1, Y_2, \dots be a sequence of i.i.d. random matrices such that the support of the common distribution is a subset of $SO(2, \mathbf{R})$. Then for each n , we can associate a real random variable $\lambda_n(\omega) \in (-1/2, 1/2]$ such that $Y_n(\omega)$ is the rotation by $2\pi\lambda_n(\omega)$, with $\lambda_n(\omega)$ i.i.d. Hence the lift Ψ satisfies $\Psi_{Y_n(\omega)}(x) = x + \lambda_n(\omega)$, for all n , and by the law of large numbers:

$$\rho(Y_1, Y_2, \dots) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \lambda_j(\omega) = \mathbb{E} [\lambda_1] \quad a.s..$$

On the other hand, since $\beta_1(\omega, x) \equiv \lambda_1(\omega)$ for all $s \in S^1$ and measure ν is the normalized Lebesgue measure on S^1 , one easily verifies formula (12). \square

EXAMPLE 3. Let Y_1, Y_2, \dots be a sequence of i.i.d. random upper triangular matrices in $Gl^+(2, \mathbf{R})$. Set $a_n(\omega)$ for the 1,1-entry of $Y_n(\omega)$, then $a_n \neq 0$ and the sequence $(a_n)_{n \geq 1}$ is i.i.d.. By induction on k the lifts Ψ_{Y_n} are such that, for k positive integer:

$$\Psi_{Y_n}(k) = \begin{cases} k & \text{if } a_n > 0 \\ k + 1/2 & \text{if } a_n < 0 \end{cases} ,$$

and

$$\Psi_{Y_n}(k + 1/2) = \begin{cases} k + 1/2 & \text{if } a_n > 0 \\ k + 1 & \text{if } a_n < 0 \end{cases} .$$

So, $\Psi_{Y_n}(k)$ “rotates” half of the circle S^1 ($\pi \circ \Psi_{Y_n}(k)$ goes to its antipode) if $a_n < 0$ and “does not rotate” (fixed) if $a_n > 0$. So one easily calculates the rotation number starting at $x = 0$:

$$\begin{aligned}\rho(Y_1, Y_2, \dots) &= \lim_{n \rightarrow \infty} \frac{\Psi_{Y_n} \circ \dots \circ \Psi_{Y_1}(0)}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \frac{1}{2} \mathbf{1}_{\{a_j < 0\}} \\ &= \frac{1}{2} \mathbb{P}[a_1 < 0] \quad a.s.\end{aligned}$$

On the other hand, if $\{e_1, e_2\}$ is the standard basis in \mathbf{R}^2 then:

$$\beta_1(\omega, e_1) = \beta_1(\omega, -e_1) = \frac{1}{2} \mathbf{1}_{\{a_1 < 0\}} \quad ,$$

and if $\delta_{(\cdot)}$ denotes the Dirac measure then:

$$\nu = \frac{1}{2} \delta_{e_1} + \frac{1}{2} \delta_{-e_1}$$

is an invariant measure on S^1 (not necessarily ergodic) and one easily verifies formula (12). □

REMARK: (*Rotation number for diffeomorphisms*) Besides the sampling theorem for continuous linear processes in R^2 (next section), another interesting application of the concept of rotation number for a sequence of random matrices is in non-linear dynamical systems. Let f be an orientation preserving diffeomorphism on \mathbf{R}^2 with a finite invariant measure μ (or in general in a 2-dimensional manifold with the support of the invariant measure contained in a neighbourhood where the tangent bundle TM is parallelizable). The rotation number of f at a point $p \in \mathbf{R}^2$, defined by the rotation number of the product of the sequence of the differential maps

$$df^{(n)}(p) = df\left(f^{(n-1)}(p)\right) \circ \dots \circ df(p) \quad ,$$

$n \geq 1$, gives the average rotation of the directions of the stable (and unstable) submanifold along the orbit of the point p . By formula (11) this number exists for μ -almost every point p , moreover it is constant in each ergodic component. It is not our purpose in this paper to go further in this non-linear analysis; it will be dealt with elsewhere.

5 Random Sampling Theorem

In this section we shall consider the discretization in time of a continuous linear cocycle on \mathbf{R}^2 , in particular, of the solution of a linear stochastic differential equation. To set up the notation let $(\theta_t)_{t \geq 0}$ be a flow of ergodic transformations on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A continuous linear (perfect) cocycle $\varphi(t, \omega)$ on \mathbf{R}^2 over θ is a map $\varphi : \Omega \times \mathbf{R}_{\geq 0} \rightarrow Gl^+(2, \mathbf{R})$ such that for all $\omega \in \Omega$:

- (i) $\varphi(0, \omega) = Id_{2 \times 2}$;
- (ii) $\varphi(t, \omega)$ is continuous on t ;
- (iii) it has the cocycle property:

$$\varphi(t + s, \omega) = \varphi(t, \theta_s(\omega)) \circ \varphi(s, \omega).$$

We deal with perfect cocycles once for every crude cocycle there exists a perfect cocycle such that they are indistinguishable, see L. Arnold and M. Scheutzow [2] or L. Arnold [1]. This cocycle generates the following random linear system on \mathbf{R}^2 :

$$x_t = \varphi(t, \omega)x_0. \quad (13)$$

We shall denote by $\xi_t(\omega)$ the induced cocycle in the unitary circle S^1 given by the radial projection of $\mathbf{R}^2 \setminus \{0\}$ onto S^1 . Associated with the process $s_t = \xi_t(\omega)s_0$ there is the real continuous angular process $\alpha_t(s_0)$ (parametrized by the initial condition $s_0 \in S^1$) such that $s_t = \exp\{i\alpha_t(s_0)\}$, *i.e.* $\alpha_t(s_0)$ is the continuous angular component of x_t . The *rotation number* $\rho(\varphi)$ of this linear system is the average angular velocity of a solution starting at $x_0 \in \mathbf{R}^2 \setminus \{0\}$:

$$\rho(\varphi) = \lim_{t \rightarrow \infty} \frac{\alpha_t(s_0)}{t},$$

when the limit exists, and if it does, it is independent of the initial point $s_0 \in S^1$.

For a fixed $T > 0$, by the cocycle property of $\varphi(t, \omega)$:

$$\varphi(nT, \omega) = \varphi(T, \theta_{(n-1)T}(\omega)) \circ \dots \circ \varphi(T, \theta_T(\omega)) \circ \varphi(T, \omega).$$

Next theorem will assure that for a quite large class of cocycles the rotation number for the product of the random matrices $\{\varphi(T, \theta_{(n-1)T}(\omega))\}_{n \geq 1}$ agrees with the rotation number $\rho(\varphi)$ of the continuous system when $T > 0$ approaches zero, up to a factor of scaling of $1/T$. We shall denote by $\mu = \nu_\omega(ds)\mathbb{P}(d\omega)$ an invariant probability measure in $S^1 \times \Omega$ for the skew-product flow in this product space induced by $\xi_t(\omega)$.

We emphasize that, although hypothesis (14) looks artificial and difficult to be verified, we prefer to state the theorem in this general formulation for cocycles and show latter (sub-sections 5.1, 5.2 and 5.3) that most of interesting dynamical systems (deterministic, real noise and stochastic respectively) satisfies naturally this hypothesis.

Theorem 5.1 (Random sampling theorem). *Consider a linear cocycle $\varphi(t, \omega)$ such that there exists the rotation number $\rho(\varphi)$ of the continuous process and*

$$\rho(\varphi) = \lim_{T \rightarrow 0} \frac{1}{T} \mathbb{E} \left[\int_{S^1} \left(\alpha_T(s) - \alpha_0(s) \right) d\nu_\omega(s) \right] \quad a.s.. \quad (14)$$

Then the rotation number $\rho(\varphi(T, \omega), \theta_T)$ of the sequence of random matrices $\{\varphi(T, \theta_{T(n-1)}(\omega))\}_{n \geq 1}$ satisfies

$$\lim_{T \rightarrow 0} \frac{1}{T} \rho(\varphi(T, \omega), \theta_T) = \rho(\varphi) \quad a.s..$$

Proof: Once stated the compatibility condition that the lifts should satisfy $\Psi_{\varphi(T, \cdot)}(0) \in (-1/2, 1/2]$ (section 3), the map $T \mapsto \Psi_{\varphi(T, \omega)}$ is continuous with respect to the uniform metric defined on $Hom^+(S^1)$ up to the stopping time

$$\sigma = \inf\{t \geq 0 : \xi_t(e_1) = -e_1\}.$$

So, fix $s_0 \in S^1$ with the corresponding initial angle $\alpha_0(s_0)$ and take a real $T > 0$, then by construction:

$$\beta_1^T(\omega, s_0) 1_{\{T \leq \sigma\}} = \left(\alpha_T(\omega) - \alpha_0(\omega) \right) 1_{\{T \leq \sigma\}}.$$

Hence by the hypothesis and the Lebesgue's convergence theorem:

$$\begin{aligned} \lim_{T \rightarrow 0} \frac{1}{T} \mathbb{E} \left[\int_{S^1} \beta_1^T(\omega, s) d\nu_\omega(s) \right] &= \\ \lim_{T \rightarrow 0} \frac{1}{T} \mathbb{E} \left[\int_{S^1} \left(\alpha_T(s) - \alpha_0(s) \right) 1_{\{T \leq \sigma\}} d\nu_\omega(s) \right] &= \rho(\varphi) \quad a.s., \end{aligned}$$

and the result follows by formula (11) and the fact that an invariant measure for the skew-product flow is also invariant for the discretized system. \square

The next three particular cases show that the theorem holds for most of the interesting linear cocycles on \mathbf{R}^2 .

5.1 The Deterministic Case

For deterministic systems it holds a more accurate result than Theorem 5.1:

Theorem 5.2 (Deterministic sampling theorem). *Consider the linear system $\dot{x} = Ax$, with A a 2×2 -matrix and let ρ be its rotation number as a continuous system. If*

$$T < \frac{1}{2\rho} ,$$

($T < \infty$ if $\rho = 0$) then

$$\frac{1}{T} \rho(\varphi_T) = \rho , \quad (15)$$

where $\varphi_T \in Gl^+(2, \mathbf{R})$ is the fundamental solution of the system at time T .

REMARK: This theorem says that if the sampling frequency is greater than 2ρ then we can retrace exactly the original frequency (rotation number) of the continuous system. For a given real signal (function) $s(t), t \in \mathbf{R}$, if f_0 is the maximum frequency of its Fourier spectrum, it is well known in the engineering literature that the whole spectrum, hence the signal, can be retraced if we sample in time this signal at a frequency greater than $2f_0$. This frequency is called the *Nyquist's rate* (see *e.g.* Papoulis [8] or Oppenheim and Schaffer [7]). If we identify each frequency in the Fourier spectrum with the corresponding rotation number of a continuous linear system on \mathbf{R}^2 , then heuristically Theorem 5.2 gives an alternative prove of the property of the Nyquist's rate.

Proof: Let λ_1, λ_2 be the eigenvalues of the matrix of coefficients A . If $\lambda_1, \lambda_2 \in \mathbf{R}$ then there exist fixed points for the continuous flows in S^1 , therefore $\rho = 0$. In this case the eigenvalues of φ_T are $e^{T\lambda_1}$ and $e^{T\lambda_2}$, hence by Proposition 4.2 $\rho(\varphi_T) = 0$ for all $T \in \mathbf{R}$.

Now assume that $\lambda_1, \lambda_2 = a \pm ib$, with $b \neq 0$. It is well known that in this case $\rho = \pm b$, with the sign depending on the orientation (see *e.g.* Arnold and San Martin [3] or San Martin [11]). Then, for an arbitrary $t \geq 0$, the eigenvalues of φ_t are $e^{t\lambda_1}, e^{t\lambda_2}$ and by Proposition 4.2 again:

$$\rho(\varphi_t) = \pm tb \pmod{1} .$$

So, for a fixed $T < 1/2\rho$ we have $\rho(\varphi_T) = \pm Tb \in (-1/2, 1/2]$, hence

$$\frac{1}{T} \rho(\varphi_T) = \pm \rho .$$

Given a fixed orientation on S^1 , say anti-clockwise, the agreement in the sign comes naturally using the ordered lifted orbit (Proposition 2.1). If $\rho > 0$, say, then the lifted orbit for ψ_{φ_T} satisfies:

$$0 < \theta_{x,n} - \theta_{x,n-1} < \frac{1}{2}$$

for all $x \in \mathbf{R}$ and $n \in \mathbb{N}$, which implies that $\rho(\varphi_T)$ is also positive. \square

REMARK: We emphasize that one of the advantages of fixing $\rho(\varphi_T)$ in the interval $(-1/2, 1/2]$ is that it makes sense to talk about “positive” or “negative” rotation number. So, the “change of the sign” when we change orientation looks more natural, besides, equality (15) makes sense without the necessity of the equivalence relation given by $(\bmod 1)$.

In the random case, because the probability of trajectories which initially rotate faster than the average rotation number is positive, the equality between the rotation number of the discrete sampled system and the continuous system only happens when we take the limit of the period T going to zero, *i.e.* in the random case it does not exist a Nyquist’s rate (*cf.* Example 4).

5.2 The Real Noise Case

Consider a linear random equation on \mathbf{R}^2 :

$$\dot{x}_t = A(\theta_t(\omega))x_t ;$$

its fundamental solution $\varphi(t, \omega)$ is a continuous linear cocycle. The continuous angular coordinate α_t of the solution x_t satisfies the random equation:

$$\dot{\alpha}_t = \langle v_t, A(\theta_t(\omega))s_t \rangle ,$$

where s_t is the radial projection of the solution x_t on the unitary circle S^1 and v_t is such that (s_t, v_t) is an orthonormal pair with anti-clockwise orientation. By the ergodic theorem the rotation number $\rho(\varphi)$ of this system satisfies:

$$\rho(\varphi) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \dot{\alpha}_r \, dr = \mathbb{E} \left[\int_{S^1} \langle v, A(\omega)s \rangle \, d\nu_\omega(s) \right] \quad a.s.,$$

where $\mu = \nu_\omega(ds)\mathbb{P}(d\omega)$ is an invariant probability measure on $S^1 \times \Omega$. So, formula (14), hence Theorem 5.1, holds in this case.

5.3 The Stochastic Case

Consider the following stochastic linear system in \mathbf{R}^2 :

$$dx_t = Ax_t dt + \sum_{i=1}^m B^i x_t \circ dW_t^i, \quad (16)$$

where A, B^1, \dots, B^m are 2×2 -matrices, $(W_t^1, \dots, W_t^m)_{t \geq 0}$ is a Brownian motion in \mathbf{R}^m with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with its natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and the integral is taken in the Stratonovich sense. This stochastic system generates a white noise linear cocycle $\varphi(t, \omega)$ (see *e.g.* Arnold [1]). The continuous angular coordinate $\alpha_t \in \mathbf{R}$ of a solution x_t with initial condition $x_0 \in \mathbf{R}^2 \setminus \{0\}$ is given by the Itô equation:

$$d\alpha_t = f(s_t) dt + \sum_{i=1}^m \langle B^i s_t, v_t \rangle dW_t^i.$$

where $f : S^1 \rightarrow \mathbf{R}$ is given by:

$$f(s) = \langle As, v \rangle + \sum_{i=1}^m \left(\frac{1}{2} \langle (B^i)^2 s, v \rangle - \langle B^i s, s \rangle \langle B^i s, v \rangle \right), \quad (17)$$

with v such that (s, v) is an orthonormal pair with anti-clockwise orientation, see Ruffino [10]. In the stochastic case the invariant measure μ on $\Omega \times S^1$ factorizes trivially a.s. $\mu = \mathbb{P} \otimes \nu$, where ν is a stationary probability measure on S^1 for the Markov processes s_t . Hence, since the average of the martingale part vanishes a.s., again, by the ergodic theorem the rotation number satisfies:

$$\rho(\varphi) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s_r) dr = \int_{S^1} f(s) d\nu(s) \quad a.s..$$

Hence the condition given by equation (14) is satisfied considering the infinitesimal generator of the processes α_t , so Theorem 5.1 also holds in this case.

We finish with an example which illustrates the fact that in the random case it does not exist the Nyquist's rate.

EXAMPLE 4: Consider the following stochastic linear system:

$$dx_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x_t dt + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x_t \circ dW_t,$$

where $x_t \in \mathbf{R}^2$ and $(W_t)_{t \geq 0}$ is a linear Brownian motion. The fundamental solution of this equation $(\varphi_t)_{t \geq 0}$ is the random linear rotation:

$$\varphi_t = \begin{pmatrix} \cos(t + W_t) & -\sin(t + W_t) \\ \sin(t + W_t) & \cos(t + W_t) \end{pmatrix} ,$$

hence the continuous rotation α_t is independent of the starting point $s \in S^1$ and is given by:

$$\alpha_t = t + W_t ;$$

therefore the rotation number

$$\rho = \lim_{t \rightarrow \infty} \frac{t + W_t}{t} = 1 \quad a.s.$$

The unique invariant probability ν in S^1 is the normalized Lebesgue measure, besides, formula (17) gives $f(s) \equiv 1$, which confirms $\rho(\varphi) = 1$ *a.s.*. For a discretization with time interval $T > 0$:

$$\beta_1^T(\omega, s) = T + W_T \pmod{1}$$

with $\beta_1^T(\omega, s) \in (-1/2, 1/2]$, and it does not depend on $s \in S^1$. The distribution of the random variable $\beta_1^T(\omega)$ corresponds to the heat kernel $P_T(T, \cdot)$ in the circle S^1 . In the figure 1 the graphics with continuous curves show its distribution for a sequence of decreasing values of T . Since the distribution of the Brownian motion on S^1 is the canonical projection of the distribution of the Brownian motion on \mathbf{R} (the universal covering space of S^1) each continuous curve in the graphic are obtained by adding up the projections of the distributions on unitary translated intervals of \mathbf{R} , represented by the dashed curves. Since in this case the rotation number $\rho(F_T, F_T \circ \theta, \dots) = \mathbb{E}[\beta_1^T]$, figure 1 makes clear the fact that

$$\frac{1}{T} \rho(F_T, F_T \circ \theta, \dots) < 1 \quad a.s. ,$$

for all $0 < T < 1/2$, so we do need the limit.

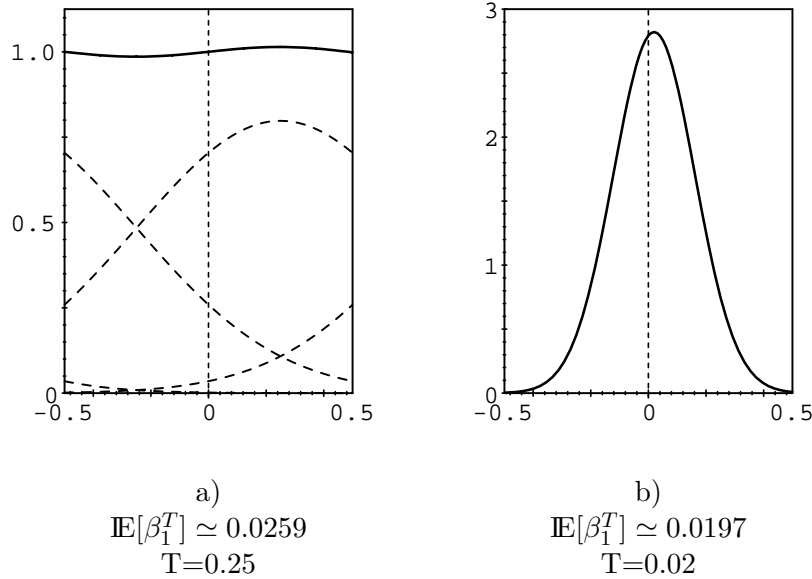


Figure 1: Distribution of β_1^T in the interval $(-1/2, 1/2]$ for decreasing values of T .

References

- [1] ARNOLD, L. – *Random Dynamical Systems*. Preliminary version, 1994.
- [2] ARNOLD, L. and SCHEUTZOW, M. – Perfect cocycles through stochastic differential equations. *Prob. Th. Rel. Fields*, **101**, p. 65–88, 1995.
- [3] ARNOLD, L. and SAN MARTIN, L.A. – A multiplicative ergodic theorem for rotation number. *J. Dynamics Differential Equations*, Vol. **1**, 1989, p. 95–119.
- [4] H. CRAUEL – Lyapunov exponents and invariant measures of stochastic systems on manifolds. In Arnold, L. and Wihstutz, V. (eds.), *Lyapunov exponents*. Lecture Notes in Math. **1186**. Springer, 1986.
- [5] FURSTENBERG, H. and KIFER, Y. – Random matrix products and measures on projective spaces. *Israel Journal of Mathematics* Vol. **46** (1983).

- [6] NITECKI, Z. – *Differentiable dynamics – an introduction to the orbit structure of diffeomorphisms*, The MIT Press, 1971.
- [7] OPPENHEIM, A.V. and SCHAFER, R.W. – *Discrete-time signal processing*. Prentice–Hall, 1989.
- [8] PAPOULIS, A. – *Signal analysis*. McGraw Hill International, 1984.
- [9] RUELLE, D. – Ergodic theory of differentiable dynamical systems. *I.H.E.S. – Publ. Math.* **50**, 1979.
- [10] RUFFINO, P. R. C. – Rotation numbers for stochastic dynamical systems. *Stochastics and Stochastics Reports*, vol. **60**, pp. 289–318, 1997.
- [11] SAN MARTIN, L.A. – **a)** Rotation numbers of differential equations: a framework in the linear case. *Relatório Técnico 05/88. Instituto de Matemática, UNICAMP, SP*, Brazil, 1988.
b) Rotation numbers in higher dimensions. *Report 199*, *Institute for Dynamical Systems*, Bremen University, Bremen, Germany, 1988.

Erratum to: A sampling theorem for rotation numbers of linear processes in \mathbf{R}^2

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1 Introduction

The purpose of these notes is to correct the arguments in the proof of the random sampling theorem (Theorem 5.1) in the stochastic case (Subsection 5.3). The problem in the original proof, where we have claimed the convergence of the limit using just Lebesgue's convergence theorem, is that, in fact, there is an extra summand (see below) which is not obvious that it goes to zero. Precisely, in lines 13-14 from bottom to top in page 184, we have that

$$\begin{aligned} \lim_{T \rightarrow 0} \frac{1}{T} \mathbf{E} \left[\int_{S^1} \beta_1^T(\omega, s_0) \, d\nu(s) \right] = \\ \lim_{T \rightarrow 0} \frac{1}{T} \mathbf{E} \left[\int_{S^1} (\alpha_T(\omega) - \alpha_0) \, 1_{T \leq \sigma} \, d\nu(s) \right] \\ + \lim_{T \rightarrow 0} \frac{1}{T} \mathbf{E} \left[\int_{S^1} (\alpha_T(\omega) - \alpha_0) \, 1_{T > \sigma} \, d\nu(s) \right]. \end{aligned}$$

Where the last summand in the right hand side was naïvely considered as zero, only based on the fact that the numerator goes to zero. Recently we have realized that depending on the (nongaussian) noise, this term may not

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vanish. Hence an extra care has to be done. Our intention here is to complete the argument, proving that, in fact, for stochastic systems (Gaussian noise) this last term does converge to zero, establishing, then the sampling theorem for this case.

Proof: Recalling our notations, for a given stochastic linear equation in \mathbf{R}^2 , the continuous angular coordinate α_t of this system satisfies

$$d\alpha_t = f(s_t)dt + \sum_{i=1}^m \langle B^i s_t, v_t \rangle dW_t^i. \quad (18)$$

where $s_t = \pi(\alpha_t)$, v_t is orthonormal to s_t with positive orientation and $f : S^1 \rightarrow \mathbf{R}$ is given by

$$f(s) = \langle As, v \rangle + \sum_{i=1}^m \left(\frac{1}{2} \langle (B^i)^2 s, v \rangle - \langle B^i s, s \rangle \langle B^i s, v \rangle \right).$$

For an initial condition $s_0 \in (-1/2, 1/2] \sim S^1$ and all $T > 0$, we have that,

$$\beta_1^T(\omega, s_0) = (\alpha_T(\omega) - \alpha_0(\omega)) + N(\omega) \quad (19)$$

for an integrable integer variable $N(\omega)$. According to our construction, $N(\omega)$ only depends on the trajectory of $\alpha_0 = 0$, or of $e_1 \in S^1$. It measures how many times this trajectory crosses its antipode ($-e_1$) in the anti-clockwise direction during the interval $[0, T]$. When $T > \tau(\omega)$, we have that $N(\omega) \neq 0$.

Hence, the proof is completed if we control the expectation $\mathbf{E}[|N|]$ and show that it goes to zero faster than T . We use boundedness on the distribution of α_t with initial condition $\alpha_0 = 0$. Let $p(t, x, y)$ be the density of the transition probability measure associated to the non-degenerate diffusions given by Equation (18). Then, there exists a constant $M > 0$ such that,

$$\frac{1}{M\sqrt{t}} e^{-M \frac{(x-y)^2}{t}} \leq p(t, x, y) \leq \frac{M}{\sqrt{t}} e^{-\frac{(x-y)^2}{Mt}}.$$

See Kusuoka and Stroock [1, 2]. Let $N^+ = \max\{N, 0\}$ and $N^- = \max\{-N, 0\}$, such that $N = N^+ - N^-$. Hence, for the positive part N^+

$$\begin{aligned} \mathbf{E}[N^+] &\leq M \int_{\alpha+1}^{\infty} ([x - (\alpha + 1)] + 1) \frac{1}{\sqrt{T}} \exp \left\{ -\frac{(x - q)^2}{MT} \right\} dx \\ &\leq M \int_{\alpha+1}^{\infty} (x - \alpha) \frac{1}{\sqrt{T}} \exp \left\{ -\frac{(x - q)^2}{MT} \right\} dx. \end{aligned}$$

And for the negative part:

$$\begin{aligned}\mathbf{E}[N^-] &\leq M \int_{-\infty}^{\alpha} ([\alpha - x] + 1) \frac{1}{\sqrt{T}} \exp \left\{ -\frac{(x - q)^2}{MT} \right\} dx \\ &\leq M \int_{-\infty}^{\alpha} (\alpha - x + 1) \frac{1}{\sqrt{T}} \exp \left\{ -\frac{(x - q)^2}{MT} \right\} dx.\end{aligned}$$

Changing variables, for $T \in (0, 1)$ we have that

$$\mathbf{E}[N^+] \leq M^{\frac{3}{2}} \int_{\frac{\alpha - q + 1}{\sqrt{MT}}}^{\infty} \left(\sqrt{M}u + q - \alpha \right) \exp \{ -u^2 \} du$$

and

$$\mathbf{E}[N^-] \leq M^{\frac{3}{2}} \int_{\frac{q - \alpha}{\sqrt{MT}}}^{\infty} \left(\sqrt{M}u + \alpha - q + 1 \right) \exp \{ -u^2 \} du$$

Hence $\mathbf{E}[N^+]$ and $\mathbf{E}[N^-]$ goes to zero when T goes to zero. Moreover, by standard calculus argument, using that $\lim_{z \rightarrow 0} \exp \{ -\frac{1}{z} \} z^{\beta} = 0$ for any exponent $\beta \in \mathbf{R}$, then finally we get that

$$\lim_{T \searrow 0} \frac{\mathbf{E}[N^+]}{T} = \lim_{T \searrow 0} \frac{\mathbf{E}[N^-]}{T} = 0.$$

□

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References

- [1] S. Kusuoka and D. Stroock – Applications of Malliavin Calculus, Part III, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **34** (1987), n. 2, 391-442.
- [2] S. Kusuoka and D. Stroock – Long time estimates for the heat kernel associated with a uniformly subelliptic symmetric second order operator, *Annals of Math.*, **127**, n. 1 (1988) 165-189.